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Interaction energy and energy release rate for mechanically loaded, thermally mismatched and interfacially cracked bimaterial bodies

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Abstract

The total stress of mechanically loaded and thermally mismatched bimaterial bodies is a sum of two stress fields. Therefore, the total Helmholtz energy consists of self-energies and interaction terms. Although the interaction strain energy vanishes (Colonnetti's theorem), the displacements purely caused by the mismatch are found to interact with the loading mechanism. The physical basis for this interaction is investigated and expressions for the energy release rate of an interfacially cracked system are derived. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Bimaterial; Energy methods; Energy release rate; Interaction; Interface; Thermal stress; Thermoelastic

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1. Introduction

With the increasing use of technical ceramics the need to bond these materials to metals also increases and several bonding technologies exist or are being developed (Klomp and de With, 1993). Often a joint is manufactured at elevated temperatures and the thermal expansion coefficients of the materials constituting the joint are usually different so that thermal residual stresses develop in and near the interface between the materials. These stresses occasionally reach values so high that the component fails during the cooling down period following the joining process. Even if the component survives the cooling process, thermal residual stresses may lead to a considerable weakening of the component, that is to say to a reduction of the, externally applied, mechanical load the component can withstand. These stresses may also have the opposite effect, i.e. toughening. Classically, the fracture load or a related quantity, for example fracture toughness, is used to characterise the fracture of a joint. However, from the point of view of materials science an energy based approach (Maugin, 1992; van der Varst and de With, 1998) is very attractive because it enables one to delineate the various processes that lead to toughening or weakening of the component. In this approach one distinguishes between mechanisms that can supply energy for crack

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growth and those that absorb energy thus decreasing the energy available for crack growth. [1] These mechanisms may be of mechanical, chemical or other nature. In this paper, however, we consider mechanisms associated with mechanical energy, that is to say with strain energy due to the imposed external traction and due to thermal mismatch, with the energy associated with a loading mechanism [2] and, specifically, interaction energies.

Crack growth starts if the crack extension force or energy release rate equals the crack resistance force (Kanninen and Popelar, 1985). The crack extension force is proportional to minus the derivative of the total potential energy of the system to the crack length. Part of this energy is the elastic or strain energy U_{el} present in the system and the other part is the potential energy U_{lm} of the loading mechanism. If mismatch stresses are present and the system is also loaded externally, the elastic energy derives from two sources; one being the mismatch of the thermal expansion coefficients and the other the external mechanical loading. In addition to that interaction may occur. Indeed, even for linear systems where the superposition principle holds stresses and strains can be superposed but for energies this is usually not possible. Generally we therefore have for the total elastic energy U_{el} the splitting $U_{\mathrm{el}} = U_{\mathrm{ex}} + U_{\mathrm{th}} + U_{\mathrm{in}}$ with U_{ex} the elastic energy caused by the external loading, U_{th} the elastic energy associated with the thermal residual stresses and U_{in} the elastic interaction energy associated with the simultaneous presence of the residual and the externally caused stress fields. With residual or internal stresses we mean stress fields that may be present in a component *that is not subjected to volume loading and without boundary conditions whatsoever*, i.e. stress fields with vanishing divergence in the interior of a volume and zero traction on the surface. Now, according to an old theorem of Colonnetti (Colonnetti, 1915; Kröner, 1958; Eshelby, 1956; Hirth and Lothe, 1992) the interaction energy for elastic fields due to external loading and due to internal stress is zero. [3] Because residual stress caused by thermal mismatch is such a system of internal stress, U_{in} is zero and one finds $U_{\mathrm{el}} = U_{\mathrm{ex}} + U_{\mathrm{th}}$. According to Eshelby (1956), a system of internal stress does, however, interact with the loading mechanism the result being that interaction terms may be present in the other contribution (potential energy of the loading mechanism) to the total energy. To describe such an interaction between loading mechanism and thermal residual stresses one would expect this interaction energy to be proportional to the applied loading, for example a force parameter f, and also proportional to the displacements v caused solely by the thermal mismatch: $U_{\mathrm{in}} \propto fv$. Eshelby's treatment (Eshelby, 1956) uses in fact these considerations. A similar approach was used by Nairn (1997). The physical basis for such an interaction energy is unclear. After all, load application usually occurs *after* joining and cooling the system meaning that the displacements v occur without any load acting on the system. So one is left with the question why a term proportional to fv should be included in the energy.

The questions raised above merit a more detailed study of the question how the external loading, thermal mismatch and temperature contribute to the total potential energy, i.e. to the energy expression from which the energy release rate is derived (Maugin, 1992).

The outline of the paper is as follows. In Section 2 the system is described. Important is that this description does not only involve shape and material properties but also its history because, prior to loading, joints have gone through a certain thermal process and it is precisely this process which leads to thermal residual stresses. Section 3 deals with the question whether an interaction between the loading mechanism and the system of internal stress due to thermal mismatch does exists and the physical basis of such an interaction. In Section 4 expressions for the energy release rate are derived and in Section 5 the most important results will be discussed.

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[1] One should be aware of the fact that there are mechanisms that can do both, for example phase transformations.

[2] A loading mechanism is a device that applies traction and it can supply or absorb energy.

[3] The original reference (Colonnetti, 1915) is quite old and not readily available. So we reproduced the theorem in Appendix A.

We use direct notation with boldface lower case roman letters, like $\mathbf{x}$, indicating vectors in three-dimensional space and boldface roman capitals, like $\mathbf{A}$, indicating second-order tensors. The term second-order tensor is used as a synonym for linear transformations from R_3 into R_3. However, the two boldface greek symbols $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ always denote the small strain tensor and the Cauchy stress tensor, respectively. Symbols like $\mathbb{L}$ denote fourth-order tensors, where a fourth-order tensor is defined as a linear transformation on the space of all second-order tensors. A single fixed Cartesian coordinate system is used with (unit) base vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. A single dot $(\cdot)$ denotes contraction over one and a double dot $(:)$ contraction over two indices. So, employing the Einstein summation convention, $\mathbf{x} \cdot \mathbf{y} = x_i y_i$, $\mathbf{A} \cdot \mathbf{x} = A_{ij} x_j$, $\mathbf{A} \cdot \mathbf{B} = A_{ik} B_{kj}$, $\mathbf{A} : \mathbf{B} = A_{ij} B_{ij}$ and $\mathbb{L} : \mathbf{A} = L_{ijkl} A_{kl}$.

2. Thermo-mechanical model of the system

2.1. Geometry, materials and loading of the joint

Consider two bodies B_1 and B_2 (boundary surfaces: ∂B_1 and ∂B_2), joined along a common part Γ (Γ_{12} = magnitude Γ) of ∂B_1 and ∂B_2 (Fig. 1). The union of the two bodies is denoted by B ($B = B_1 \cup B_2$) and the external surface of the joined bodies by S ($S = (\partial B_1 \cup \partial B_2) \setminus \Gamma$). Each body is homogeneous (anisotropy is allowed) but they differ with respect to thermal and mechanical properties. The joint operates at temperature T and a traction system $\mathbf{t}$ acts on the surface part S_t. No volume loading is present. The traction system is assumed to be prescribed and self-equilibrating (total external force and external moments are zero) and not to act on the crack faces. So, the part S_r of S is traction-free and includes the crack faces. The system as a whole is not supported in any way. This case also includes a system for which the applied traction is not self-equilibrating provided the system is supported in a statically determinate manner.

2.2. Thermal and processing history

A joint is generally made at a temperature T_m higher than the operating temperature T ($T \leqslant T_m$). Prior to bonding at T_m, both bodies are assumed to be stress-free and the surface parts at which the bodies will be

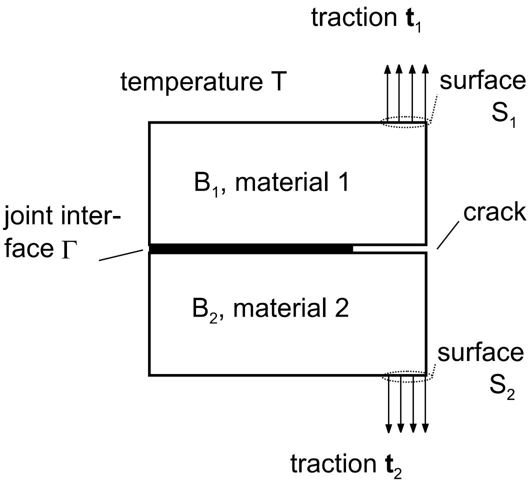

Fig. 1. Bimaterial joint under operating conditions.

joined are plane. After manufacturing the joint the whole system is cooled down to its operating temperature T. Often joints are made by brazing or solid state bonding in which a thin interlayer (a third material) has been used. Generally the interlayer material is rather ductile because one wants to prevent the build-up of residual stresses through plastic flow of the interlayer material. Mechanically this means that thermal residual stresses only build up below a certain temperature. Typically one would expect this temperature to be approximately equal to the recrystallisation temperature of the interlayer material (Klomp, 1993) because for many metals the stress needed to deform the material plastically is quite small above the recrystallisation temperature. To mimick this behaviour for a bimaterial joint (no interlayer), we assume that thermal residual stresses exist only for temperatures below some temperature T_r, and that during cooling down to T no stress relief takes place. So, at $T = T_r$ the bodies are stress-free and this state is used as a stress-free reference state. The joint can then be modelled by requiring that material points adjacent to the interface occupy the same position for all temperatures $T \leqslant T_r$.

After producing the joint (joining and cooling) several other processes may follow (for example surface grinding). Because the focus is here on thermal residual stresses we assume that these processes do not lead to other residual stress systems. This means that bodies of the same shape and material, not bonded together but otherwise subjected to the same thermal and processing history, remain stress-free; in the absence of bonding, stress-free states exist in each of the bodies for all $T \leqslant T_r$.

The load is applied at the operating temperature T and this temperature is assumed to be constant afterwards.

2.3. *Thermo-mechanical modelling*

2.3.1. *General part*

We consider only slow processes and situations with no body forces or heat sources and a uniform temperature distribution at any time (no inertia effects, decoupled thermo-mechanical equations). As variables we use the strains and the temperature so that, as is custom in thermodynamics, the Helmholtz energy is the thermodynamic potential to be used in the analysis.

To study the interaction between loading mechanism and joint the bimaterial body and the loading mechanism are considered as one single, mechanically isolated system. Exchange of thermal energy with its environment, a heat bath with temperature $T_B = T$, is possible. At equilibrium the Helmholtz energy H of the system is minimal if the temperature of the system is equal to that of the heat bath $T = T_B$ (Ericksen, 1991). The Helmholtz energy of the total system is

$$H = H_s + H_b + H_f, \tag{1}$$

with H_s the contribution of the surfaces and the interface, H_b that of the bulk and H_f the contribution of the loading mechanism.

2.3.2. *Joint constraint and decomposition of the displacement*

Mechanically, the joint is a kinematical constraint on the total *displacement* $\mathbf{u}$ of the material points adjacent to the interface from their reference position $\mathbf{x}$ at T_r. The total displacement $\mathbf{u}$ – with respect to its stress-free reference position $\mathbf{x}$ at the reference temperature T_r – is written as the sum of three independent contributions

$$\mathbf{u} = \mathbf{r} + \mathbf{v} + \mathbf{w}. \tag{2}$$

The first, $\mathbf{r}$, is due to cooling of non-joined and non-loaded bodies from the reference temperature T_r to the current temperature T. The second, $\mathbf{v}$, is caused by enforcement of the joint constraint at the current

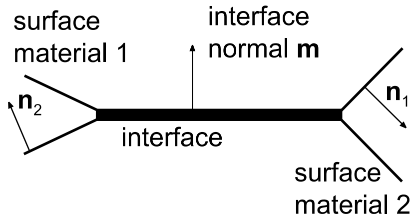

Fig. 2. Normals on the bodies and the interface.

temperature (but prior to load application) and the third, $\mathbf{w}$, is caused purely by application of the traction to the cooled and joined bodies. The joint constraint is [4]

$$[\mathbf{u}(\mathbf{x},T)] = \lim_{\xi\to 0}(\mathbf{u}(\mathbf{x}+\xi\mathbf{m}\cdot T) - \mathbf{u}(\mathbf{x}-\xi\mathbf{m},T)) = 0, \quad \mathbf{x}\in\Gamma \tag{3}$$

with $\mathbf{m}$ the interface normal (Fig. 2). Using the decomposition (2), this constraint is found to split in two independent constraints

$$[\mathbf{r}+\mathbf{v}] = \mathbf{0} \quad \text{and} \quad [\mathbf{w}] = \mathbf{0}, \quad \mathbf{x}\in\Gamma. \tag{4}$$

2.3.3. Surfaces and the interface

The contribution of the surfaces and the interface is

$$H_\mathrm{s} = \int_S \gamma(\mathbf{x})\,\mathrm{d}S + \int_\Gamma \gamma_{12}\,\mathrm{d}S \quad \text{with } \gamma(\mathbf{x}) = \begin{cases} \gamma_1, & \mathbf{x}\in B_1, \\ \gamma_2, & \mathbf{x}\in B_2. \end{cases} \tag{5}$$

The Helmholtz surface energy of the non-bonded surface regions is γ and the interfacial Helmholtz energy is γ_{12}. Although the surface and interface energy generally depends on the deformation (Gurtin and Murdoch, 1975; Zangwill, 1990), we neglect this dependency.

2.3.4. Bulk of the material

The material behaviour of each material is assumed to be linear elastic and homogeneous (Erdogan, 1975) and we consider the small displacement gradient approximation.

Let $\boldsymbol{\epsilon} = (\nabla\mathbf{u} + (\nabla\mathbf{u})^\mathrm{T})/2$ be the strain tensor and $\theta = T - T_\mathrm{r}$ the temperature difference. The expression for H_b is found by considering the Helmholtz volume energy density as a quadratic function of $\boldsymbol{\epsilon}$ and θ and using that for $\theta = 0$ and $\boldsymbol{\epsilon} = 0$ the materials are in a stress-free state (Landau and Lifshitz, 1986). The bulk Helmholtz energy is then found to be a *functional* of $\mathbf{u}$ and a function of θ

$$H_\mathrm{b}(\mathbf{u}) = H_\mathrm{b}^{(\mathrm{r})} + \int_B (\psi(\mathbf{u}) - \theta\mathbf{M} : \boldsymbol{\epsilon}(\mathbf{u}))\,\mathrm{d}V, \quad \left(\psi = \frac{1}{2}\boldsymbol{\epsilon}(\mathbf{u}) : \mathbb{L} : \boldsymbol{\epsilon}(\mathbf{u})\right) \tag{6}$$

with $H_\mathrm{b}^{(\mathrm{r})}$ a temperature, volume and material dependent reference value and ψ the strain energy density. $\mathbb{L}$ is the fourth-order tensor of elastic constants with the usual symmetry properties (Malvern, 1969) and $\mathbf{M}$ the symmetric second-order stress–temperature tensor. For further use we also need the (symmetric) tensor of thermal expansion coefficients $\boldsymbol{\alpha}$ defined by $\boldsymbol{\alpha} = \mathbb{L}^{-1} : \mathbf{M}$. All these tensors depend on the position but are piecewise constant, i.e.

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[4] A jump of some quantity c across the interface is indicated by square brackets, i.e. $[c] = c_1 - c_2$ and the average of these values by a tilde $\tilde{c} = (c_1 + c_2)/2$.

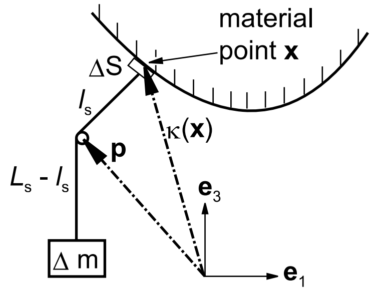

Fig. 3. Simple loading mechanism.

$$\mathbb{L}(\mathbf{x}), \mathbf{M}(\mathbf{x}), \boldsymbol{\alpha}(\mathbf{x}) = \begin{cases} \mathbb{L}_1, \mathbf{M}_1, \boldsymbol{\alpha}_1, & \mathbf{x} \in B_1, \\ \mathbb{L}_2, \mathbf{M}_2, \boldsymbol{\alpha}_2, & \mathbf{x} \in B_2. \end{cases} \tag{7}$$

2.3.5. The loading mechanism

To study the interaction it is necessary to model the loading mechanism in more detail. Consider first a simple loading mechanism (Fig. 3) where the load is applied by the mass Δm attached to the body (at the surface material point $\mathbf{x}$) by means of an inextensible massless and ideally flexible string of length L_s running over a massless pulley positioned at the point $\mathbf{p}$. The current position of the attachment point $\mathbf{x}$ is $\boldsymbol{\kappa}$. The pulley diameter is very small compared to $l_s = ||\mathbf{p} - \boldsymbol{\kappa}||$. The direction of the load is $(\mathbf{p} - \boldsymbol{\kappa})/l_s$, and – assuming that the load is distributed over the surface element ΔS – the magnitude is $\omega \Delta S$ with $\omega = g\Delta m/\Delta S$ (g constant of gravity) leading to the traction $\mathbf{t} = \omega(\mathbf{p} - \boldsymbol{\kappa})/l_s$. Let $\mathbf{h}$ be the current position of the mass Δm, i.e. $\mathbf{h} = \mathbf{p} - (L_s - l_s)\mathbf{e}_3$, and the Helmholtz energy (potential energy) $H_m = g\mathbf{h} \cdot \mathbf{e}_3 \Delta m$; an expression that can be rewritten as

$$H_m = \omega(\mathbf{p} \cdot \mathbf{e}_3 - (L_s - ||\mathbf{p} - \boldsymbol{\kappa}||))\Delta S. \tag{8}$$

This simple model is in fact a special version of a class of a continuously distributed dead weight loading systems characterised by the potential energy

$$H_{fm} = \int_{S_t} \omega(\mathbf{x})(\mathbf{e}_3 \cdot \mathbf{p}(\mathbf{x}) - (L_s(\mathbf{x}) - ||\mathbf{p}(\mathbf{x}) - \boldsymbol{\kappa}(\mathbf{x})||))\,\mathrm{d}S, \tag{9}$$

where $\mathbf{x}$ labels the points of the surface region S_t, $\boldsymbol{\kappa}(\mathbf{x})$ is the current position of these points and ω, L_s and $\mathbf{p}$ are given functions of $\mathbf{x}$.

To simplify the model and apply it to the present case, H_{fm} is expanded in a Taylor series around a base line function $\boldsymbol{\kappa}_0(\mathbf{x}) = \mathbf{x}$ and truncated after the linear term. So the Helmholtz energy H_f is

$$H_f(\boldsymbol{\kappa}) = H_{fm}(\mathbf{x}) - \int_{S_t} \mathbf{t} \cdot (\boldsymbol{\kappa} - \mathbf{x})\,\mathrm{d}S \tag{10}$$

with $\mathbf{t}(\mathbf{x}) = \omega(\mathbf{x})(\mathbf{p}(\mathbf{x}) - \boldsymbol{\kappa}_0(\mathbf{x}))/||\mathbf{p}(\mathbf{x}) - \boldsymbol{\kappa}_0(\mathbf{x})||$ the traction. Without a loss in generality, the base line function $\boldsymbol{\kappa}_0$ is chosen as $\boldsymbol{\kappa}_0(\mathbf{x}) = \mathbf{x}$ ($\mathbf{x} \in S_t$). Writing $\boldsymbol{\kappa}(\mathbf{x}) = \boldsymbol{\kappa}_r(\mathbf{x}) + \mathbf{w}(\mathbf{x})$, where the function $\boldsymbol{\kappa}_r$ will be specified later, the Helmholtz energy of the loading mechanism is

$$H_f = \begin{cases} H_f^{(r)} & \text{before load application,} \\ H_f^{(r)} + H_{li}(\mathbf{w}) & \text{after load application} \end{cases} \tag{11}$$

with $H_f^{(r)}$ and H_{li} defined by

$$H_{\mathrm{f}}^{(\mathrm{r})} = H_{\mathrm{fm}}(\mathbf{x}) - \int_{S_{\mathrm{t}}} \mathbf{t} \cdot (\boldsymbol{\kappa}_{\mathrm{r}} - \mathbf{x})\,\mathrm{d}S, \qquad H_{\mathrm{li}}(\mathbf{w}) = -\int_{S_{\mathrm{t}}} \mathbf{t} \cdot \mathbf{w}\,\mathrm{d}S. \tag{12}$$

2.3.6. The basic equations

The decomposition of the total displacements was made for analytical purposes. However, each contribution corresponds in fact to a possible physical process. The result of Ericksen as given previously then implies that **r**, **v** and **w** can be found from a variational principle (see also Washizu (1983)) whereby the Helmholtz energy, relevant to the considered case, is to be minimised. As the case may be, minimisation is constrained to ensure that the bodies remain connected.

The first contribution, **r**, is found by unconstrained minimisation of the Helmholtz energy $H_{\mathrm{b}}(\mathbf{r})$ of the bulk because at this stage the loading mechanism is not yet coupled to the joint. This leads to $\mathbb{L} : \boldsymbol{\epsilon}(\mathbf{r}) = \theta\mathbf{M}$ with solution (Carlson, 1984)

$$\mathbf{r}(\mathbf{x}) = \theta\boldsymbol{\alpha}(\mathbf{x})(\mathbf{x} - \mathbf{x}_0) + \mathbf{a}(\mathbf{x}). \tag{13}$$

Here, $\mathbf{x}_0$ and $\mathbf{a}(\mathbf{x})$ are arbitrary constant vectors and strainless rigid body displacements, respectively, that both may exhibit jumps across the interface.

Using the solution (13) for **r**, the second contribution, **v**, is found by constrained (see the first of Eq. (4)) minimisation of $H_{\mathrm{b}}(\mathbf{r} + \mathbf{v})$. The constraint induces an internal or residual stress $\boldsymbol{\sigma}_{\mathrm{c}}$ and

$$\boldsymbol{\sigma}_{\mathrm{c}} = \mathbb{L} : \boldsymbol{\epsilon}(\mathbf{v}), \qquad \operatorname{div} \boldsymbol{\sigma}_{\mathrm{c}} = 0, \quad \mathbf{x} \in B, \tag{14}$$

$$\boldsymbol{\sigma}_{\mathrm{c}} \cdot \mathbf{n} = 0, \quad \mathbf{x} \in S = S_{\mathrm{r}} \cup S_{\mathrm{t}}, \tag{15}$$

$$[\mathbf{v}] = -[\mathbf{r}], \qquad [\boldsymbol{\sigma}_{\mathrm{c}} \cdot \mathbf{m}] = \mathbf{0}, \quad \mathbf{x} \in \Gamma. \tag{16}$$

The bulk energy now splits in two terms $H_{\mathrm{b}}(\mathbf{r} + \mathbf{v}) = H_{\mathrm{b}}(\mathbf{r}) + H_{\mathrm{th}}(\mathbf{v})$ with

$$H_{\mathrm{b}}(\mathbf{r}) = H_{\mathrm{b}}^{(\mathrm{r})} - \frac{1}{2}\theta^2 \int_B (\boldsymbol{\alpha} : \mathbb{L} : \boldsymbol{\alpha})\,\mathrm{d}V, \qquad H_{\mathrm{th}}(\mathbf{v}) = \int_B \psi(\mathbf{v})\,\mathrm{d}V. \tag{17}$$

Obviously, H_{th} is the elastic self-energy or strain energy associated with the combination of thermal mismatch and joint constraint.

Finally, the third contribution, **w**, is determined by constrained (see the second of Eq. (4)) minimisation of the Helmholtz energy $H_{\mathrm{b}}(\mathbf{r} + \mathbf{v} + \mathbf{w}) + H_{\mathrm{f}}(\boldsymbol{\kappa}_{\mathrm{r}}, \mathbf{w})$ of bodies and loading mechanism considered as one single system. Just prior to load application the points of S_{t} occupy the position $\mathbf{x} + \mathbf{r} + \mathbf{v}$ and load application adds **w**. Coupling of the loading mechanism must be performed without doing work. [5] So, the reference function $\boldsymbol{\kappa}_{\mathrm{r}}$ is chosen as $\boldsymbol{\kappa}_{\mathrm{r}} = \mathbf{x} + \mathbf{r} + \mathbf{v}$ and this leads to

$$H_{\mathrm{f}} = H_{\mathrm{fm}}(\mathbf{x}) + H_{\mathrm{li}}(\mathbf{r} + \mathbf{v}) + H_{\mathrm{li}}(\mathbf{w}). \tag{18}$$

Note that the $H_{\mathrm{li}}(\mathbf{w})$ is the classic potential energy associated with the traction system. The load induces a stress $\boldsymbol{\sigma}_{\mathrm{f}}$ and

$$\boldsymbol{\sigma}_{\mathrm{f}} = \mathbb{L} : \boldsymbol{\epsilon}(\mathbf{w}), \qquad \operatorname{div} \boldsymbol{\sigma}_{\mathrm{f}} = 0, \quad \mathbf{x} \in B, \tag{19}$$

$$\boldsymbol{\sigma}_{\mathrm{f}} \cdot \mathbf{n} = 0, \quad \mathbf{x} \in S_{\mathrm{r}}, \tag{20}$$

$$\boldsymbol{\sigma}_{\mathrm{f}} \cdot \mathbf{n} = \mathbf{t}, \quad \mathbf{x} \in S_{\mathrm{t}}, \tag{21}$$

$$[\mathbf{w}] = 0, \qquad [\boldsymbol{\sigma}_{\mathrm{f}} \cdot \mathbf{m}] = \mathbf{0}, \quad \mathbf{x} \in \Gamma. \tag{22}$$

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[5] In terms of the simple model from Section 2.3.5 this means that the mass must be positioned in such a way that after cooling and joining the string can be attached without slack or additional lifting of the mass so that load application occurs by removing the support of the mass.

The total stress is $\boldsymbol{\sigma} = \boldsymbol{\sigma}_c + \boldsymbol{\sigma}_f$ and the total Helmholtz energy of the bulk, $H_b(\mathbf{u})$, is now found to consist of three separate terms

$$H_b(\mathbf{u}) = H_b(\mathbf{r}) + H_{th}(\mathbf{v}) + H_{tr}(\mathbf{w}), \quad \text{with } H_{tr} = \int_B \psi(\mathbf{w})\,\mathrm{d}V. \tag{23}$$

H_{tr} is the elastic self-energy or strain energy associated with the traction induced stress fields $\boldsymbol{\sigma}_f$. Eq. (23) contains no interaction elastic energy. The interaction elastic energy density $\boldsymbol{\epsilon}(\mathbf{w}) : \mathbb{L} : \boldsymbol{\epsilon}(\mathbf{v})$ is not locally zero but its integrated form vanishes on account of Colonnetti's theorem.

3. Interaction with the loading mechanism

Collecting all previous results it found that the total Helmholtz energy of the system is

$$H = H_s + H_b(\mathbf{r}) + H_{th}(\mathbf{v}) + H_{tr}(\mathbf{w}) + H_{li}(\mathbf{w}) + H_{fm}(\mathbf{x}) + H_{li}(\mathbf{r} + \mathbf{v}). \tag{24}$$

All contributions are neatly separated with H_s being the contribution of the interface and the surfaces, $H_b(\mathbf{r})$ the bulk energy of the unloaded and freely shrunk materials, $H_{th}(\mathbf{v})$ the elastic strain energy due to the mismatch and the interface constraint, $H_{tr}(\mathbf{w})$ the classic strain energy caused by the loading, $H_{li}(\mathbf{w})$ the classic potential energy of the loading mechanism and, finally, the last two terms represent the reference value of the energy of the loading mechanism. Part of this energy is the interaction energy. To see this, note that [6]

$$H_{li}(\mathbf{r} + \mathbf{v}) = -\int_{S_t} (\mathbf{r} + \mathbf{v}) \cdot \mathbf{t}\,\mathrm{d}S = -\theta \int_B \boldsymbol{\alpha} : \boldsymbol{\sigma}_f\,\mathrm{d}V, \tag{25}$$

$$\int_{S_t} \bar{\mathbf{r}} \cdot \mathbf{t}\,\mathrm{d}S = \theta\tilde{\boldsymbol{\alpha}} : \int_B \boldsymbol{\sigma}_f\,\mathrm{d}V \qquad \left(\bar{\mathbf{r}} = \frac{\theta}{2}(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2) \cdot \mathbf{x} = \theta\tilde{\boldsymbol{\alpha}} \cdot \mathbf{x}\right). \tag{26}$$

Writing $H_{li}(\mathbf{r} + \mathbf{v})$ as $H_{li}(\mathbf{r} + \mathbf{v} - \bar{\mathbf{r}}) + H_{li}(\bar{\mathbf{r}})$ suggests that the part

$$H_{in} = H_{li}(\mathbf{r} + \mathbf{v} - \bar{\mathbf{r}}) = -\int_{S_t} \mathbf{t} \cdot (\mathbf{r} + \mathbf{v} - \bar{\mathbf{r}})\,\mathrm{d}S \tag{27}$$

$$H_{in} = H_{li}(\mathbf{r} + \mathbf{v} - \bar{\mathbf{r}}) = -\theta \int_B (\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}) : \boldsymbol{\sigma}_f\,\mathrm{d}V \tag{28}$$

of $H_{li}(\mathbf{r} + \mathbf{v})$ is the interaction energy. H_{in} is not only proportional to the loading and to the temperature difference θ but it also vanishes if no mismatch is present, i.e. if $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2$. The remaining part of $H_{li}(\mathbf{r} + \mathbf{v})$, $\theta\tilde{\boldsymbol{\alpha}} : \int \boldsymbol{\sigma}_f\,\mathrm{d}V$, does not vanish if the expansion properties are equal although it is proportional to loading and temperature difference. This energy term is not associated with mismatch and it is, unlike H_{in}, independent of the crack length as can be seen from the left-hand side of Eq. (26).

It is instructive to consider Eq. (28) for the case of isotropic materials having equal elastic but different expansion properties. One then obtains

$$H_{in} = -\tfrac{1}{2}\theta(\alpha_1 - \alpha_2)K(\Delta V_1 - \Delta V_2) \tag{29}$$

with α_i the expansion coefficients, K the bulk modulus and ΔV_i the volume changes of the bodies purely caused by the applied loading.

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[6] The derivation is omitted for reasons of space but Eq. (25) can be derived using $\theta\boldsymbol{\alpha} = \boldsymbol{\epsilon}(\mathbf{r})$, Colonnetti's theorem, the divergence theorem, mechanical equilibrium, the continuity properties of $\mathbf{r} + \mathbf{v}$ and $\boldsymbol{\sigma}_f$ at the interface, the fact that S_r is traction-free and that $\mathbf{t} = \boldsymbol{\sigma}_f \cdot \mathbf{n}$ on S_t. Eq. (26) can be derived in a similar way.

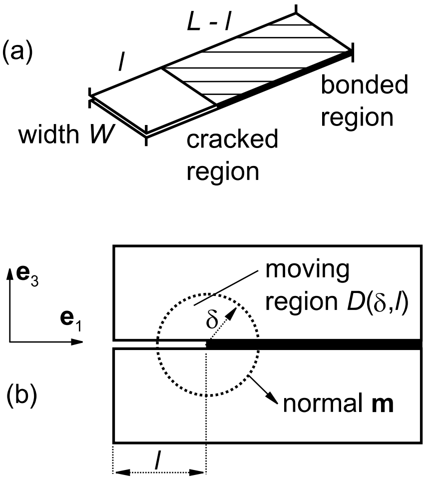

Fig. 4. Straight interface crack, uniform width W (a) and cylindrical region D (surface ∂D, $\mathbf{m}$ outward normal to D) moving with crack tip (b).

4. Energy release rate

To proceed consider a situation of a joint having a uniform width W and an interfacial crack of length l with a straight front (Fig. 4a). The energy release rate G is

$$G = -\frac{1}{W}\frac{\mathrm{d}H}{\mathrm{d}l}\bigg|_{\mathrm{ttf}} = G_{\mathrm{s}} + G_{\mathrm{th}} + G_{\mathrm{me}} + G_{\mathrm{in}} \tag{30}$$

in which the subscript ttf indicates that during differentiation temperature and tractions are to be kept fixed. G_{s} is the contribution [7] of the surfaces and the interface, G_{th} the contribution of the thermal residual stress caused by the thermal mismatch, G_{me} the classic mechanical part of the energy release rate and G_{in} the interaction term. Specifically

$$G_{\mathrm{s}} = \gamma_{12} - (\gamma + \gamma_2), \tag{31}$$

$$G_{\mathrm{th}} = -\frac{1}{W}\frac{\partial}{\partial l}\int_B \psi(\mathbf{v})\,\mathrm{d}V, \tag{32}$$

$$G_{\mathrm{me}} = \frac{1}{W}\int_{S_{\mathrm{t}}}\frac{\partial \mathbf{w}}{\partial l}\cdot \mathbf{t}\,\mathrm{d}S - \frac{\partial}{\partial l}\int_B \psi(\mathbf{w})\,\mathrm{d}V, \tag{33}$$

$$G_{\mathrm{in}} = \int_{S_{\mathrm{t}}}\frac{\partial \mathbf{v}}{\partial l}\cdot \mathbf{t}\,\mathrm{d}S. \tag{34}$$

For G_{me}, G_{th} and G_{in} crack tip integrals can be derived. One has to account, however, for the fact that differentiation (to crack length) and integration over the volume B do not commute. The approach [8] used in crack tip analysis is to introduce first a small region D (∂D is the surface of D and $\mathbf{m}$ the outward normal on ∂D) moving with the crack tip, next to differentiate and afterwards to let this region shrink to zero (Fig. 4b). This leads to

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[7] In materials science $-G_{\mathrm{s}}$ is called the work of adhesion J_{ad}.

[8] For a clear exposition of this approach we refer to Gurtin (1979). For reasons of space only the result is given.

$$G_{\mathrm{th}} = \frac{1}{W}\int_{\mathrm{ct}} \left(\psi(\mathbf{v})\,\mathbf{e}_1 \cdot \mathbf{m} + \frac{\partial \mathbf{v}}{\partial l} \cdot \boldsymbol{\sigma}_{\mathrm{c}} \cdot \mathbf{m}_i\right) \mathrm{d}S, \tag{35}$$

$$G_{\mathrm{me}} = \frac{1}{W}\int_{\mathrm{ct}} \left(\psi(\mathbf{w})\,\mathbf{e}_1 \cdot \mathbf{m} + \frac{\partial \mathbf{w}}{\partial l} \cdot \boldsymbol{\sigma}_{\mathrm{f}} \cdot \mathbf{m}\right) \mathrm{d}S, \tag{36}$$

$$G_{\mathrm{in}} = \frac{1}{W}\int_{\mathrm{ct}} \left(\boldsymbol{\epsilon}(\mathbf{v}) : \boldsymbol{\sigma}_{\mathrm{f}}\,\mathbf{e}_1 \cdot \mathbf{m} + \frac{\partial \mathbf{v}}{\partial l} \cdot \boldsymbol{\sigma}_{\mathrm{f}} \cdot \mathbf{m} + \frac{\partial \mathbf{w}}{\partial l} \cdot \boldsymbol{\sigma}_{\mathrm{c}} \cdot \mathbf{m}\right) \mathrm{d}S. \tag{37}$$

The subscript ct indicates crack tip integration, i.e. $\int_{\mathrm{ct}} = \lim \delta \to 0 \int_{\partial D(\delta,l)}$. Expressions (35)–(37) become in fact J-integrals if we assume – as is usually done for homogeneous bodies (Gurtin and Podio-Guidugli, 1996; Maugin, 1992; Nguyen, 1981) – that near the crack tip and for a crack progressing in the x-direction $-\partial\mathbf{v}/\partial x$ and $-\partial\mathbf{w}/\partial x$ are the dominant terms of $\partial\mathbf{v}/\partial l$ and $\partial\mathbf{w}/\partial l$, respectively. In the limit of vanishing δ it is then allowed to replace in expressions (35)–(37) the operator $\partial/\partial l$ by $-\partial/\partial x$ thus leading to the familiar formulae for the J-integral.

In the experimental situation proportional loading is often used, i.e.

$$\mathbf{t} = f\mathbf{q}, \quad \mathbf{w} = f\hat{\mathbf{w}}(\mathbf{x}, l), \quad \boldsymbol{\sigma}_f = f\hat{\boldsymbol{\sigma}}_{\mathrm{f}}(\mathbf{x}, l), \tag{38}$$

with f the load factor. Also

$$\mathbf{v} = \theta\hat{\mathbf{v}}(\mathbf{x}, l) + \text{rigid body displacements independent of } l. \tag{39}$$

Next, define the mechanical, mismatch and interaction compliance functions $C_{\mathrm{me}}(l)$, $C_{\mathrm{th}}(l)$ and $C_{\mathrm{in}}(l)$, respectively, by

$$C_{\mathrm{me}}(l) = \int_B \psi(\hat{\mathbf{w}})\,\mathrm{d}V, \quad C_{\mathrm{th}}(l) = \int_B \psi(\hat{\mathbf{v}})\,\mathrm{d}V, \tag{40}$$

$$C_{\mathrm{in}}(l) = \int_B (\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}) : \hat{\boldsymbol{\sigma}}_{\mathrm{f}}\,\mathrm{d}V. \tag{41}$$

Using that at mechanical equilibrium $2H_{\mathrm{tr}}(\mathbf{w}) = \int_{S_{\mathrm{t}}} \mathbf{t} \cdot \mathbf{w}\,\mathrm{d}S$ gives

$$G_{\mathrm{me}} = \frac{1}{2W}\frac{\mathrm{d}C_{\mathrm{me}}}{\mathrm{d}l} f^2, \qquad G_{\mathrm{th}} = -\frac{1}{2W}\frac{\mathrm{d}C_{\mathrm{th}}}{\mathrm{d}l}\theta^2, \qquad G_{\mathrm{in}} = \frac{1}{W}\frac{\mathrm{d}C_{\mathrm{in}}}{\mathrm{d}l} f\theta. \tag{42}$$

Now we have a global representation because the sum $G_{\mathrm{me}} + G_{\mathrm{th}} + G_{\mathrm{in}}$ is a quadratic form in the global parameters f and θ and the coefficients of this quadratic form are proportional to the derivatives of the compliance functions. Denoting the two eigenvalues of this form by ω_1 and ω_2 it is found that

$$\omega_1 + \omega_2 = \frac{1}{W}\left[\frac{\mathrm{d}C_{\mathrm{me}}}{\mathrm{d}l} - \frac{\mathrm{d}C_{\mathrm{th}}}{\mathrm{d}l}\right], \qquad \omega_1\omega_2 = \frac{-1}{W^2}\left[\frac{\mathrm{d}C_{\mathrm{me}}}{\mathrm{d}l}\frac{\mathrm{d}C_{\mathrm{th}}}{\mathrm{d}l} + \left(\frac{\mathrm{d}C_{\mathrm{in}}}{\mathrm{d}l}\right)^2\right]. \tag{43}$$

The magnitudes of the eigenvalues determine whether the sum $G_{\mathrm{th}} + G_{\mathrm{in}} + G_{\mathrm{me}}$ is positive definite or not.

5. Discussion

The part H_{in} of the total Helmholtz energy is indeed an interaction energy because Eq. (28) shows that the absence of either a load, a temperature difference or a difference in thermal expansion properties of the materials leads to $H_{\mathrm{in}} = 0$. Eqs. (27) and (28) for the interaction energy are equivalent, yet reflect different points of view. Eq. (27) directs attention to the surfaces S_{t} where the loading mechanism is attached to the bodies, whereas Eq. (28) focuses on the volume. The former equation is therefore the key to the physical basis of the interaction energy, namely a part of the reference value for the energy of the loading mechanism.

Eq. (28) for H_{in} is similar to a result in the theory of point defects. For isotropic materials with V_i and α_i the volumes and coefficients of linear expansion, respectively, p_i the volume averaged hydrostatic pressure induced by the applied traction, the interaction energy, Eq. (28), becomes

$$H_{\mathrm{in}} = 3\theta \sum_{i=1}^{2} (\alpha_i - \tilde{\alpha}) V_i p_i. \tag{44}$$

This result resembles the expression $\delta V p(\boldsymbol{\xi})$ for the energy associated with the interaction between a point defect (located at $\mathbf{x} = \boldsymbol{\xi}$ and leading to a global volume change δV) and an external loading system locally exerting the pressure $p(\mathbf{x})$ (Eshelby, 1956).

The sum of G_{me}, G_{in} and G_{th} is a quadratic form, the sign of which is determined by the magnitudes of the derivatives of the three compliance functions (cf. Eq. (43)). The sign of G_{me} is positive because crack growth leads to an increase in mechanical compliance, i.e. $\mathrm{d}C_{\mathrm{me}}/\mathrm{d}l > 0$. For the $\mathbf{v}$ displacements crack growth is equivalent to relaxing the interface constraint. Since the $\mathbf{v}$ displacements are determined by constrained minimisation of a non-negative functional the actual value of H_{th}, that is to say the value of this functional attained at the minimum, decreases if the constraint is progressively relaxed and H_{th} reaches its absolute minimum value zero if $l = L$ (no constraint at all). Therefore we conclude that generally $G_{\mathrm{th}} > 0$ and from this that $\mathrm{d}C_{\mathrm{th}}/\mathrm{d}l < 0$. Combining Eq. (43) with the inequalities $\mathrm{d}C_{\mathrm{me}}/\mathrm{d}l > 0$ and $\mathrm{d}C_{\mathrm{th}}/\mathrm{d}l < 0$ shows that $\omega_1 + \omega_2 > 0$. Consequently, at least one of the two eigenvalues is positive because both are real. One then finds from the second of Eq. (43) that both eigenvalues are positive, i.e. $G_{\mathrm{me}} + G_{\mathrm{th}} + G_{\mathrm{in}} > 0$ whenever

$$\frac{\mathrm{d}C_{\mathrm{in}}}{\mathrm{d}l}^2 < -\frac{\mathrm{d}C_{\mathrm{me}}}{\mathrm{d}l}\frac{\mathrm{d}C_{\mathrm{th}}}{\mathrm{d}l}. \tag{45}$$

Note that it is not the sign of $\mathrm{d}C_{\mathrm{in}}/\mathrm{d}l$ that is important but its magnitude. Considering that the stress intensity factor for each fracture mode is proportional to $c_f f + c_\theta \theta$ (c_f and c_θ dependent on the considered mode) and that the usual energy release rate, that is to say $G_{\mathrm{me}} + G_{\mathrm{th}} + G_{\mathrm{in}}$, is proportional to the sum of the squares of the stress intensity factors for the three modes (Nairn, 1997) one finds that $G_{\mathrm{me}} + G_{\mathrm{th}} + G_{\mathrm{in}}$ is always positive. So, the inequality (45) is always met.

The quadratic form for $G_{\mathrm{me}} + G_{\mathrm{th}} + G_{\mathrm{in}}$ does not tell us to which mode each term contributes. If one knows in advance that all terms contribute to one single mode, say mode I, then G_{me}, G_{th} and G_{in} are related (Nairn, 1997). Vanishing G_{in} combined with non-zero G_{me} and G_{th} indicates a situation where loading and residual stress field operate on different modes. Consider for example a double cantilever beam specimen where the two materials have the same shape and elastic properties but differ in thermal expansion coefficient. Application of Eq. (29) and using symmetry arguments then shows that for this system $H_{\mathrm{in}} = 0$ (and therefore also $G_{\mathrm{in}} = 0$) for mechanical mode I and mode III loading. Since the elastic energy due to the mismatch H_{th} is the same irrespective whether mechanical loading is mode I, II or III we conclude that in this example G_{th} contributes to mode II.

Energy terms depending on the temperature difference θ, the crack length l and the mismatch can be used to determine how the entropy $S = -\partial H/\partial T$ of the system (bodies plus loading mechanism) changes if crack growth occurs.

$$\frac{\partial S}{\partial l} = \begin{cases} W\partial G_{\mathrm{s}}/\partial\theta, & \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2, \\ W\partial(G_{\mathrm{s}} + G_{\mathrm{th}} + G_{\mathrm{in}})/\partial\theta, & \boldsymbol{\alpha}_1 \neq \boldsymbol{\alpha}_2. \end{cases} \tag{46}$$

A crack can be considered as a macroscopic defect and its size, the crack length l, is thermodynamically an internal variable. The Helmholtz energy H of the system (surface, bulk and loading device) is a potential for the thermodynamic force [9] F associated with l. Or, $F = -\partial H/\partial l$ where all other macroscopic parameters (temperature and boundary conditions) have to be kept constant in the differentiation process. In our case this leads to

$$\frac{F}{W} = G = -\frac{1}{W}\frac{\mathrm{d}H}{\mathrm{d}l}\bigg|_{\mathrm{ttf}}. \tag{47}$$

In mechanics one usually does not include the surface term H_s in the definition of G. If, however, one wants to study combined mechanical and chemical phenomena this term is needed. Apart from this surface influence on G, our energy release rate is the same as encountered in mechanics because the Helmholtz energy $H - H_\mathrm{s}$ is, not counting crack length independent terms, equal to the elastic potential.

Provided dissipation due to plastic and viscous effects can be neglected, the second law of thermodynamics demands that $F\mathrm{d}l/\mathrm{d}t \geqslant 0$ or, considering that at the onset of crack growth $\mathrm{d}l/\mathrm{d}t \geqslant 0$, that $G \geqslant 0$. Rearranging the last inequality leads to a Griffith criterion

$$G_\mathrm{me} + G_\mathrm{th} + G_\mathrm{in} \geqslant -G_\mathrm{s} = J_\mathrm{ad} > 0, \tag{48}$$

where J_ad is the (positive) work of adhesion. Since this equation is derived from thermodynamics' second law (an inequality) it becomes clear why the Griffith criterion is basically one of necessity. Because $G_\mathrm{th} \geqslant 0$ we see that the mismatch stresses always tend to weaken a component. This effect may be mitigated or enhanced by the interaction term because G_in may be of either sign.

6. Conclusions

In this paper a traction loaded thermally mismatched bimaterial body was considered. Each material is homogeneous and linear elastic but anisotropy is allowed. Expressions for the Helmholtz energy and the energy release rate for a traction-free interface crack were derived.

The Helmholtz energy was found to contain a term H_in which is indeed an interaction energy because the absence of either a load, a temperature difference or a difference in thermal expansion properties of the materials renders this term zero. The origin of this energy is actually a correction of the reference value of the energy of the loading mechanism one needs to include so as to account for the shape change the interface constraint brings about to an otherwise unloaded body. When after load application the interface crack starts to grow these displacements also change because interface crack growth is equivalent to relaxing the constraints for these displacements. The traction system obviously may perform work against these displacement changes and this work is in addition to the work resulting from displacement changes purely caused by the increase of structural compliance.

The sum of G_me, G_in and G_th is for proportional loading a positive quadratic form in θ and the load parameter f. This expression does not tell us to which fracture mode (I, II, III) each term contributes. However, when material properties, loading and geometry combine in such a way that the interaction term vanishes one can conclude that mechanical and mismatch stress fields operate on different modes. Alternatively, in case the interaction does not vanish the two stress fields operate on at least one common fracture mode.

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[9] For an extensive review of internal variable theory we refer to Maugin and Muschik (1994). For a treatment similar to ours, see Rice (1978). In recent literature the thermodynamic force F is called configurational force (Gurtin and Podio-Guidugli, 1996) or material force (Maugin, 1998).

The mismatch stresses always tend to weaken a component because the corresponding energy release rate is always positive. The presence of an interaction term, however, may enhance or mitigate this effect because the energy release rate associated with it may be of either sign .

Finally, we want to point out that the procedure of the analysis remains the same if the crack is not along the interface but away from it.

Appendix A. The theorem of Colonnetti

Let $\boldsymbol{\sigma}$ be an internal stress field in a body B, i.e. a stress field for which $\text{div}\,\boldsymbol{\sigma} = \mathbf{0}$ in B and $\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$ on the surface S of B (no volume forces and boundary conditions). Now let $\mathbf{u}$ be an arbitrary displacement field ($\boldsymbol{\epsilon}$ the strain associated with $\mathbf{u}$) caused by applying boundary conditions. Starting from mechanical equilibrium (Einstein summation convention employed) $\sigma_{ij,j} = 0$ for the internal field, one finds $u_i\sigma_{ij,j} = (u_i\sigma_{ij})_{,j} - \epsilon_{ij}\sigma_{ij} = 0$. Integrating and applying the divergence theorem obtains

$$\int_V \boldsymbol{\epsilon} : \boldsymbol{\sigma}\,\mathrm{d}V = \int_S \mathbf{u} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}\,\mathrm{d}S = 0. \tag{A.1}$$

So, the energy associated with the interaction between the internal stress field and that caused by arbitrary boundary conditions is zero.

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